2/16/26 Exam 1 next Wed 2/25.

Covers all of ch1, ch2. None of ch3. (Although we'll start Ch 3 Wednesday.)

<u>No</u> calculators! Cell phones should be put away.

Today: some odds & ends to finish Ch 2

Last time: we worked out

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} -9 & 2 & 2 \\ 3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ie they're inverses of each other) so if we have the linear system

$$\begin{aligned} x + 2y + 2z &= 5 \\ 3x + 5y + 6z &= 4 \\ 2x + 4y + 3z &= 3 \end{aligned}$$

then there's a unique solution:

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

so

$$\begin{bmatrix} -9 & 2 & 2 \\ 3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 & 2 & 2 \\ 3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -31 \\ 11 \\ 7 \end{bmatrix} \qquad \text{(you can check)}$$

Similarly the homog linear system

$$\begin{aligned} x + 2y + 2z &= 0 \\ 3x + 5y + 6z &= 0 \\ 2x + 4y + 3z &= 0 \end{aligned}$$

has the unique solution

$$\begin{bmatrix} -9 & 2 & 2 \\ 3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

What if

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 5 & 6 \\ 4 & 7 & 8 \end{bmatrix} ?$$

Consider $A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Row reduce

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 3 & 5 & 6 & 3 \\ 4 & 7 & 8 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & -1 & 0 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Can't row reduce to I_3 $\rightsquigarrow$ rank$(A) = 2$ so

A is not invertible

and $A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is inconsistent.

But what about $A\vec{x} = \vec{0}$?

Same process reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{ie} \quad \begin{aligned} x + 2z &= 0 \\ y &= 0 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x &= -2t \\ y &= 0 \\ z &= t \end{aligned}$$

It <u>is</u> consistent, with infinitely many solutions

Remark (a) if A and B are invertible $n \times n$ matrices (so A^{-1}, B^{-1} both exist) then AB is invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Check: $(AB)(B^{-1}A^{-1}) = I_n$ (associativity)

(b) When $n = 2$ there's a convenient shortcut to determine if A is invertible, and if so, how to find it.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $ad - bc$ is the determinant of A, denoted $\det(A)$.

A is invertible if and only if $\det(A) \neq 0$. In that case

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Check)

Remark We'll say a lot more about determinants later!

Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is not invertible since $\det(A) = 0$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible since $\det(A) = -2$

then $A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ check ...

Interesting detail:

<u>Thm</u> (2.4.8)

Let A, B be $n \times n$ matrices such that $BA = I_n$. Then

(a) A, B are both invertible

(b) $A^{-1} = B$ and $B^{-1} = A$

(c) $AB = I_n$ (reverse order)

pf/

(a) consider the lin. system $A\vec{x} = \vec{0}$.

WTS it has only the trivial solution. (so $\text{rank}(A) = n$)

$$\underbrace{B(A\vec{x})}_{= \vec{x}} = B\vec{0} = \vec{0}$$

(b) Since the inverse is unique, $B = A^{-1}$ and $A = B^{-1}$

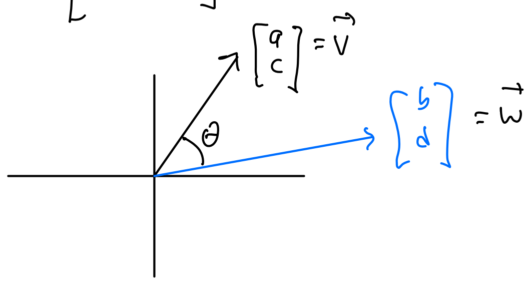
(c) immediate since $B = A^{-1}$

Last thing from §2.4: geometric interpretation of the determinant in the 2×2 case.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Look at the column vectors

Let $\vec{v} = \begin{bmatrix} a \\ c \end{bmatrix}$ $\vec{w} = \begin{bmatrix} b \\ d \end{bmatrix}$.



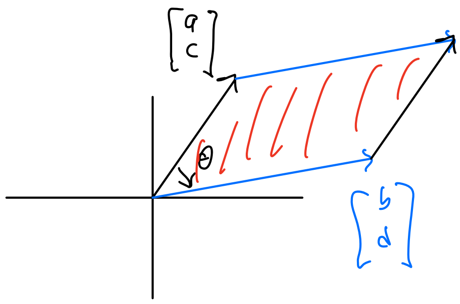
Let θ be the angle from $\vec{v}$ to $\vec{w}$, so

$$-\pi < \theta \leq \pi$$

then

<u>Fact</u> $\det A = ad - bc = \|\vec{v}\|\,\|\vec{w}\| \sin\theta$

$=$ area of the parallelogram determined by $\vec{v}$ and $\vec{w}$



(you can see the details in the book.)

Next time: start ch 3 (not on exam)